Quantum Mechanics as a Classical Theory X: Quantization in Generalized Coordinates

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Abstract

In this tenth paper of the series we aim at showing that our formalism, using the Wigner-Moyal Infinitesimal Transformation together with classical mechanics, endows us with the ways to quantize a system in any coordinate representation we wish. This result is necessary if one even think about making general relativistic extensions of the quantum formalism. Besides, physics shall not be dependent on the specific representation we use and this result is necessary to make quantum theory consistent and complete.

1 Overview

The present paper is the tenth part of a series of papers on the mathematical and epistemological foundations of quantum theory. The papers of this series may fit into one (or more) of four rather different categories of interest.

They may be considered as reconstruction papers, where the existing formalism is derived within the (classical) approach defined by the use of Wigner-Moyal's Infinitesimal Transformation. Into such division we might put papers I,II,III[1, 2, 3] and also papers VI,VII and VIII[6, 7, 8].

There are also papers where we resize the underlying epistemology to fit the (purely classical) mathematical developments of the theory. In these cases, alleged purely quantum effects are reinterpreted on classical grounds—from the epistemological perspective. Into this category we might put papers III,VI,VII and IX[3, 6, 7, 9].

Pertaining to another class of papers we have those trying to expand the applicability of quantum theory to other fields of investigation. It is remarkable that such a task was taken with paper II[2], where a quantum theory (of one particle) that takes into account the effects of gravity was developed, and with

papers IV and V[4, 5], where this generalized relativistic quantum theory was applied to show that it predicts particles with negative masses.

The last category is the one where we try to investigate the boundaries of quantum theory—there where it seems to give unsatisfactory answers, both from the mathematical and epistemological points of view. We might cite paper IX as one example where the important question of operator formation was discussed at length. Papers belonging to this category try to remove from the theory some of its formal problems, as was the case with paper IX, or misinterpretations, as was the case with paper III where the problem of non-locality was investigated.

The present paper pertains to this last class of interest and is particularly related with formal problems. We have been taught, since our very introduction to the study of quantum mechanics, that, for quantizing a system, we shall first write its *classical* hamiltonian in *cartesian coordinates*. This quantization may be mathematically represented by

$$\mathbf{p} \rightarrow -i\hbar\nabla \ , \ \mathbf{x} \rightarrow x, \tag{1}$$

where $\mathbf{p}$ is the momentum and $\mathbf{x}$ the coordinate, and by the transformation

$$H(\mathbf{p}, \mathbf{x}) \rightarrow H(-i\hbar\nabla, x). \tag{2}$$

It is only after this quantization has been performed that we may change to another system of coordinates, distinct form the cartesian one[10].

This seems to be a terrible problem, although not seemed as such by many of us, since physics is supposed to treat any (mathematical) system of coordinates on the same grounds. If not for this reason, one may wonder about the future of general relativistic extensions of a theory that *needs* the flat cartesian system of coordinates to exist. Within a relativistic theory this need seems to be a scandal that denies *from the very beginning* any such extension.

One may find in the literature [12, 13, 14] some trials to overcome these difficulties, but even these approaches are permeated with additional suppositions as in ref. [12, 13] where the author has to postulate that the total quantum-mechanical momentum operator p_{q_i} corresponding to the generalized coordinate q_i is given by

$$p_{q_i} = -i\hbar\frac{\partial}{\partial q_i} \tag{3}$$

and has also to write the classical hamiltonian (the kinetic energy term) as

$$H = \frac{1}{2m}\sum_{i,k} p^*_{q_i} g^{ik} p_{q_k} \tag{4}$$

whatever be the complex conjugate of the classical momentum *function*.

These approaches seem to be rather unsatisfactory for we would like to derive our results using only first principles, without having to add more postulates to the theory.

This problem appears because quantum mechanics, as developed in textbooks, is not a theory with a clearly discernible set of axioms[11]. Indeed, the rules (1) and (2) above are part of the fuzzy set of axioms one could append to it.

We have developed a completed axiomatic (classical) version of quantum mechanics which, we expect, does not depend on the specific set of coordinates used.

The aim of this paper is to show that our expectations are confirmed by the mathematical formalism.

We will show in the second section how to quantize a hamiltonian with a central potential in spherical coordinates using only the three axioms we have already postulated, now written in this coordinate system. This will serve as an illustrative example of how quantization in generalized coordinates shall be done.

The third section will aim at generalizing the previous particular approach to any set of orthogonal generalized coordinates; that is, to show how to quantize in such coordinates.

2 Spherical Coordinates: an example

We begin by rewriting our three axioms[1] in the appropriate coordinate system:

Axiom 1: Newtonian particle mechanics is valid for all particles which constitute the systems composing the *ensemble*;

Axiom 2: For an *ensemble* of isolated systems the joint probability density function is a constant of motion:

$$\frac{dF(r,\theta,\phi,p_r,p_\theta,p_\phi;t)}{dt}=0; \tag{5}$$

Axiom 3: The Wigner-Moyal Infinitesimal Transformation, defined as

$$\rho(\mathbf{r}-\frac{\delta\mathbf{r}}{2},\mathbf{r}+\frac{\delta\mathbf{r}}{2};t)=\int F(r,\theta,\phi,p_r,p_\theta,p_\phi;t)\exp(\frac{i}{\hbar}\mathbf{p}\cdot\delta\mathbf{r})d^3p \tag{6}$$

may be applied to represent the dynamics of the system in terms of functions $\rho(\mathbf{r},\delta\mathbf{r};t)$.

Using expression (5) (Liouville's equation), we may write

$$\frac{\partial F}{\partial t}+\{H,F\}=0, \tag{7}$$

where H is the hamiltonian and $\{,\}$ is the classical Poisson bracket. By means of the hamiltonian, written in spherical coordinates

$$H=\frac{1}{2m}\left[p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\phi^2}{r^2\sin^2(\theta)}\right]+V(r), \tag{8}$$

we find the Liouville equation

$$\frac{\partial F}{\partial t}+\frac{p_r}{m}\frac{\partial F}{\partial r}+\frac{p_\theta}{mr^2}\frac{\partial F}{\partial\theta}+\frac{p_\phi}{mr^2\sin^2(\theta)}\frac{\partial F}{\partial\phi}-$$

$$-\left[\frac{\partial V}{\partial r}-\frac{p_\theta^2}{mr^3}-\frac{p_\phi^2}{mr^3\sin^2(\theta)}\right]\frac{\partial F}{\partial p_r}+\frac{p_\phi^2}{mr^2\sin^2(\theta)}\cot(\theta)\frac{\partial F}{\partial p_\theta}=0. \quad (9)$$

As a means of writing the Wigner-Moyal Transformation in spherical coordinates, we note that

$$\delta\mathbf{r}=\delta r\widehat{\mathbf{r}}+r\delta\theta\widehat{\theta}+r\sin(\theta)\delta\phi\widehat{\phi} \quad (10)$$

where $(\widehat{\mathbf{r}},\widehat{\theta},\widehat{\phi})$ are the unit normals, and

$$\mathbf{p}=p_r\widehat{\mathbf{r}}+\frac{p_\theta}{r}\widehat{\theta}+\frac{p_\phi}{r\sin(\theta)}\widehat{\phi}, \quad (11)$$

giving

$$\mathbf{p}\cdot\delta\mathbf{r}=\delta r\cdot p_r+\delta\theta\cdot p_\theta+\delta\phi\cdot p_\phi. \quad (12)$$

Using the relation between the unit normals in cartesian $(\mathbf{i},\mathbf{j},\mathbf{k})$ and spherical $(\widehat{\mathbf{r}},\widehat{\theta},\widehat{\phi})$ coordinates

$$\left\{\begin{array}{l}\widehat{\mathbf{r}}=\mathbf{i}\sin(\theta)\cos(\phi)+\mathbf{j}\sin(\theta)\sin(\phi)+\mathbf{k}\cos(\theta)\\ \widehat{\theta}=\mathbf{i}\cos(\theta)\cos(\phi)+\mathbf{j}\cos(\theta)\sin(\phi)-\mathbf{k}\sin(\theta)\\ \widehat{\phi}=-\mathbf{i}\sin(\phi)+\mathbf{j}\cos(\phi)\end{array}\right. , \quad (13)$$

we find the following relation between the momenta:

$$\left\{\begin{array}{l}p_x=p_r\sin(\theta)\cos(\phi)+(p_\theta/r)\cos(\theta)\cos(\phi)-(p_\phi/r)(\sin(\phi)/\sin(\theta))\\ p_y=p_r\sin(\theta)\sin(\phi)+(p_\theta/r)\cos(\theta)\sin(\phi)+(p_\phi/r)(\cos(\phi)/\sin(\theta))\\ p_z=p_r\cos(\theta)-(p_\theta/r)\sin(\theta)\end{array}\right. . \quad (14)$$

The Jacobian relating the two infinitesimal volume elements

$$dp_xdp_ydp_z=\|J\|_p\,dp_rdp_\theta dp_\phi \quad (15)$$

is given by

$$\|J\|_p=\frac{1}{r^2\sin(\theta)}. \quad (16)$$

It is now possible to rewrite expression (6) as

$$\rho(\mathbf{r}-\frac{\delta\mathbf{r}}{2},\mathbf{r}+\frac{\delta\mathbf{r}}{2};t)=\int F(\mathbf{r},\mathbf{p};t)e^{\frac{i}{\hbar}(\delta r\cdot p_r+\delta\theta\cdot p_\theta+\delta\phi\cdot p_\phi)}\frac{dp_rdp_\theta dp_\phi}{r^2\sin(\theta)}. \quad (17)$$

With equation (9) and expression (17) at hands we may find the equation satisfied by the density function $\rho(\mathbf{r},\delta\mathbf{r};t)$ in exactly the same way as was previously done for cartesian coordinates[1]—it is noteworthy that now we have the jacobian (16) that will change slightly the appearance of this equation terms.

After some straightforward calculations we arrive at

$$-\frac{\hbar^2}{m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho}{\partial(\delta r)}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\rho}{\partial(\delta\theta)}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\rho}{\partial\phi\partial(\delta\phi)}\right]+$$

$$+\frac{\hbar^2}{m}\left[\frac{\delta r}{r^3}\frac{\partial^2\rho}{\partial(\delta\theta)^2}+\frac{\delta r}{r^3\sin^2(\theta)}\frac{\partial^2\rho}{\partial(\delta\phi)^2}+\frac{\delta\theta\cot(\theta)}{r^2\sin^2(\theta)}\frac{\partial^2\rho}{\partial(\delta\phi)^2}\right]+\delta r\frac{\partial V}{\partial r}\rho=i\hbar\frac{\partial\rho}{\partial t}. \tag{18}$$

To go from this equation to the equation for the amplitudes we may write

$$\rho(\mathbf{x}-\frac{\delta\mathbf{x}}{2},\mathbf{x}+\frac{\delta\mathbf{x}}{2};t)=\psi^\dagger(\mathbf{x}-\frac{\delta\mathbf{x}}{2};t)\psi(\mathbf{x}+\frac{\delta\mathbf{x}}{2};t) \tag{19}$$

and also write

$$\psi(\mathbf{x};t)=R(\mathbf{x};t)\exp\left(iS(\mathbf{x};t)/\hbar\right) \tag{20}$$

in cartesian coordinates, for example. We then expand expression (19) around the infinitesimals quantities to get, until second order,

$$\rho(\mathbf{x},\delta\mathbf{x};t)=\left\{R^2+\frac{R}{4}\sum_{i,j=1}^{3}\delta x_i\delta x_j\frac{\partial^2 R}{\partial x_i\partial x_j}-\frac{1}{4}\sum_{i,j=1}^{3}\delta x_i\delta x_j\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j}\right\}\cdot$$

$$\cdot\exp\left[\frac{i}{\hbar}\left(\delta x\frac{\partial S}{\partial x}+\delta y\frac{\partial S}{\partial y}+\delta z\frac{\partial S}{\partial z}\right)\right], \tag{21}$$

where $x_i, i=1,2,3$ implies x,y,z, and use the relations

$$\left\{\begin{array}{l}\partial_x=\sin(\theta)\cos(\phi)\partial_r+(1/r)\cos(\theta)\cos(\phi)\partial_\theta-(1/r)(\sin(\phi)/\sin(\theta))\partial_\phi\\ \partial_y=\sin(\theta)\sin(\phi)\partial_r+(1/r)\cos(\theta)\sin(\phi)\partial_\theta+(1/r)(\cos(\phi)/\sin(\theta))\partial_\phi\\ \partial_z=\cos(\theta)\partial_r-(1/r)\sin(\theta)\partial_\theta\end{array}\right., \tag{22}$$

where ∂_u is an abbreviation of $\partial/\partial u$, to write the density function in spherical coordinates as

$$\rho(\mathbf{r},\delta\mathbf{r};t)=\left\{R^2+\frac{R}{4}\left[\delta r^2\frac{\partial^2 R}{\partial r^2}+\delta\theta^2\left(\frac{\partial^2 R}{\partial\theta^2}+r\frac{\partial R}{\partial r}\right)+\right.\right.$$

$$+\delta\phi^2\left(\frac{\partial^2 R}{\partial\phi^2}+r\sin^2(\theta)\frac{\partial R}{\partial r}+\cos(\theta)\sin(\theta)\frac{\partial R}{\partial\theta}\right)+$$

$$2\delta r\delta\theta\left(\frac{\partial^2 R}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial R}{\partial\theta}\right)+2\delta r\delta\phi\left(\frac{\partial^2 R}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial R}{\partial\phi}\right)-$$

$$+2\delta\theta\delta\phi\left(\frac{\partial^2 R}{\partial\theta\partial\phi}-\cot(\theta)\frac{\partial R}{\partial\phi}\right)\Bigg]-\frac{1}{4}\sum_{i,j=1}^{3}\delta x_i\delta x_j\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j}\Bigg\}\cdot$$

$$\cdot\exp\left[\frac{i}{\hbar}\left(\delta r\frac{\partial S}{\partial r}+\delta\theta\frac{\partial S}{\partial\theta}+\delta\phi\frac{\partial S}{\partial\phi}\right)\right], \qquad (23)$$

where now $x_i, i=1,2,3$ means r,θ,ϕ.

The next step is to take expression (23) into equation (18) and to collect the zeroth and first order terms in the infinitesimals[1] to get, as usual[1], the equations (written in spherical coordinates)

$$\delta\mathbf{r}\cdot\frac{\partial}{\partial\mathbf{r}}\left[\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+V(r)-\frac{\hbar^2}{2mR}\nabla^2 R\right]=0 \qquad (24)$$

and

$$\frac{\partial R^2}{\partial t}+\nabla\cdot\left(\frac{R^2}{m}\nabla S\right)=0. \qquad (25)$$

Equation (25) is the continuity equation while equation (24) is the same as writing

$$\delta\mathbf{r}\cdot\frac{\partial}{\partial\mathbf{r}}\left[\left(\widehat{H}\psi-i\hbar\frac{\partial\psi}{\partial t}\right)\frac{1}{\psi}\right]=0, \qquad (26)$$

where the ψ is as given in expression (20) and the hamiltonian operator H is given by

$$\widehat{H}=\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]+V(r). \qquad (27)$$

In this sense, we may say, since the infinitesimals are all independent, that we have derived the Schrödinger equation written as

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]\psi+$$

$$+V(r)\psi=i\hbar\frac{\partial\psi}{\partial t}, \qquad (28)$$

where the constant coming from expression (26) might be appended in the right hand term above and reflects a mere definition of a new reference energy level.

We thus have quantized the system using only spherical coordinates from the very beginning as was our interest to show.

In the next section we will generalize this result to any set of orthogonal coordinate systems.

[1]this cumbersome exercise was performed using algebraic computation.

3 Orthogonal Coordinates

In this case we have the transformation rules

$$x_\alpha = x_\alpha(u_1, u_2, u_3) \ ; \ \alpha = 1, 2, 3, \tag{29}$$

where x_α are the coordinates written in some system (not necessarily the cartesian one), u_i are the coordinates in the new system and the differential line element is given by[15]

$$d\mathbf{r} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3, \tag{30}$$

where the $\mathbf{e}_i$ are the unit normals in the new (u_1, u_2, u_3)-coordinate system and

$$h_i \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{31}$$

The momenta (p_1, p_2, p_3) canonically conjugate to the u-coordinate system are given by

$$\mathbf{p} = m\left(h_1 \frac{du_1}{dt}\mathbf{e}_1 + h_2 \frac{du_2}{dt}\mathbf{e}_2 + h_3 \frac{du_3}{dt}\mathbf{e}_3\right) = $$
$$= \left(\frac{p_1}{h_1}\mathbf{e}_1 + \frac{p_2}{h_2}\mathbf{e}_2 + \frac{p_3}{h_3}\mathbf{e}_3\right), \tag{32}$$

or

$$p_i = m h_i^2 \frac{du_i}{dt}, \tag{33}$$

such that

$$\mathbf{p} \cdot \delta\mathbf{u} = p_1 \delta u_1 + p_2 \delta u_2 + p_3 \delta u_3. \tag{34}$$

The Hamiltonian may be written as[16]

$$H = \frac{1}{2m}\left[\frac{p_1^2}{h_1^2} + \frac{p_2^2}{h_2^2} + \frac{p_3^2}{h_3^2}\right] + V(\mathbf{u}) \tag{35}$$

and the Liouville equation becomes

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{3} \frac{p_i^2}{m h_i^2}\frac{\partial F}{\partial u_i} + \sum_{j=1}^{3}\left[\sum_{i=1}^{3}\left(\frac{p_i^2}{m h_i^3}\frac{\partial h_i}{\partial u_j}\right) - \frac{\partial V}{\partial u_j}\right]\frac{\partial F}{\partial p_j} = 0. \tag{36}$$

Since the coordinate transformation (29) is a special type of canonical transformation, we shall have the Jacobian of the momentum transformation given by

$$\|J\|_p = \frac{1}{h_1 h_2 h_3}, \tag{37}$$

for the Jacobian of the coordinate transformation is

$$\|J\|_u = h_1 h_2 h_3 \tag{38}$$

and the infinitesimal phase space volume element is a canonical invariant[17].

With the density function given by

$$\rho\left(\mathbf{u}-\frac{\delta\mathbf{u}}{2},\mathbf{u}+\frac{\delta\mathbf{u}}{2};t\right)=\int F(\mathbf{u},\mathbf{p};t)\exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\delta\mathbf{u}\right)\frac{dp_1dp_2dp_3}{h_1h_2h_3}, \tag{39}$$

it is straightforward to find the differential equation it satisfies as

$$-i\hbar\frac{\partial\rho}{\partial t}-\frac{\hbar^2}{m}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial(\delta u_1)}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial(\delta u_2)}\right)+\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial(\delta u_3)}\right)\right]-\sum_{i,j=1}^{3}\frac{\delta u_i}{h_j^3}\frac{\partial h_j}{\partial u_i}\frac{\partial^2\rho}{\partial(\delta u_i)^2}\right\}+\sum_{i=1}^{3}\delta u_i\frac{\partial V}{\partial u_i}\rho=0. \tag{40}$$

The reader may easily verify that, with the coordinate transformation given by

$$\left\{\begin{array}{l} x=r\sin\theta\cos\phi\\ y=r\sin\theta\sin\phi\\ z=r\cos\theta\end{array}\right. ;\left\{\begin{array}{l} h_1=1\\ h_2=r\\ h_3=r\sin\theta\end{array}\right. , \tag{41}$$

we recover the result of equation (18).

We now write the density function as

$$\rho(\mathbf{u}-\frac{\delta\mathbf{u}}{2},\mathbf{u}+\frac{\delta\mathbf{u}}{2};t)=\psi^{\dagger}(\mathbf{u}-\frac{\delta\mathbf{u}}{2};t)\psi(\mathbf{u}+\frac{\delta\mathbf{u}}{2};t), \tag{42}$$

with the, generally complex, amplitudes written as

$$\psi(\mathbf{x};t)=R(\mathbf{u};t)\exp\left(iS(\mathbf{u};t)/\hbar\right) \tag{43}$$

and expand these amplitudes until second order in the infinitesimal parameter $\delta\mathbf{u}$, to find (until second order)

$$\rho(\mathbf{u},\delta\mathbf{u};t)=\left\{R^2+\frac{R}{4}\left[\sum_{i,j=1}^{3}\delta u_i\delta u_j\left(\frac{\partial^2R}{\partial u_i\partial u_j}-\sum_{k=1}^{3}\Gamma_{ij}^{k}\frac{\partial R}{\partial u_k}\right)\right]-\right.$$
$$\left.-\frac{1}{4}\left[\sum_{i,j=1}^{3}\delta u_i\delta u_j\frac{\partial R}{\partial u_i}\frac{\partial R}{\partial u_j}\right]\right\}\cdot\exp\left(\frac{i}{\hbar}\sum_{i=1}^{3}\delta u_i\frac{\partial S}{\partial u_i}\right), \tag{44}$$

where Γ is the Christoffel Symbol[18]. The reader may verify that expression (44) gives the correct (23) result when expressed in spherical coordinates.

We then insert this expression into equation (40) and collect the zeroth and first order terms on the infinitesimals[2] to get the equations (written in general orthogonal coordinates)

$$\delta\mathbf{u}\cdot\frac{\partial}{\partial\mathbf{u}}\left[\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S\right)^2+V(\mathbf{u})-\frac{\hbar^2}{2mR}\nabla^2R\right]=0 \tag{45}$$

[2]Again, algebraic computation was used throughout.

and

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(\frac{R^2}{m} \nabla S \right) = 0. \tag{46}$$

Equation (46) is the continuity equation while equation (45) is the same as writing

$$\delta \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{u}} \left[\left(\widehat{H}\psi - i\hbar \frac{\partial \psi}{\partial t} \right) \frac{1}{\psi} \right] = 0, \tag{47}$$

where the ψ is as shown in expression (43) and the hamiltonian operator $\widehat{H}$ is given by

$$\widehat{H} = \frac{-\hbar^2}{2m} \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_2} \right) \right.$$
$$\left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right] + V(\mathbf{u}). \tag{48}$$

Because the infinitesimals are all independent, the term inside brackets in equation (47) must vanish identically. This allows us to say that we have derived the Schrödinger equation written as

$$\frac{-\hbar^2}{2m} \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_2} \right) \right.$$
$$\left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right] \psi + V(\mathbf{u})\psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{49}$$

where the constant coming from expression (47) might be appended in the right hand term above and reflects a mere new definition of the reference energy level—or a new definition of the function S by adding this constant factor.

We thus have quantized the system using only general orthogonal coordinates from the very beginning as was our aim to show. The extension of this result for non-orthogonal coordinate systems is straightforward and will not be done here.

4 Conclusions

We have shown that the process of quantization is coordinate system independent. This result does not bring anything new to the machinery of quantum theory, since its formal apparatus for application on computational problems remains untouched.

However, when showing this coordinate system independence, we also show that the formalism is in conformity with our expectations that *physics shall not depend on the way we choose to represent it.*

Although this seems to be sterile from the point of view of calculations, it gives the formalism coherence and wideness. Coherence for the reasons explained above and wideness for now it is possible to justify any trial to find a general relativistic extension of this formalism.

These results may also be seen as another confirmation of our guesses about the classical nature of quantum theory.

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